Abstract—The objective of this paper is the derivation in a closed form of the channel estimation mean square error (MSE) expression in the time-frequency domain for OFDM systems, providing a general tool that can be applied to all linear channel estimation algorithms. The proposed method can be used to identify the best channel estimation algorithms, and to compare the different pilot symbol positioning strategies within an OFDM frame. Thanks to these features the proposed method can be used as a support to the system design.

I. INTRODUCTION

Aiming at providing very high data-rates as well as QoS and spectral efficiency in the field of wireless communications, Orthogonal Frequency Division Multiplexing (OFDM) has been recently selected as the transmission technique which is generally proposed for the upcoming systems [1] [2].

In order to properly deal with time and frequency selectivity, advanced channel estimators are required which allow an effective equalization at the receiver side. As a result, the channel estimation plays a fundamental role in the equalization procedure and determines the receiver performance [3] [4].

As for the estimation techniques, pilot based approaches are generally adopted for their reduced computational burden and remarkable performance: overviews of some of the pilot aided channel estimation techniques can be found in [5] and in [6].

The first objective of this paper is the derivation of the channel estimation mean square error (MSE) in the time-frequency domain for all the main linear channel estimation algorithms, so allowing a general comparison between them.

It is worth noticing that the MSE expression can be achieved for any distribution of the pilot symbols according to a general lattice within the OFDM frame: therefore, as a second result, we propose a theoretical tool which can be used to define the best pilot symbol patterns for any OFDM transmission format, by comparing the different positioning strategies of the pilot symbols.

Notations: In the following, matrix variables are indicated with boldface uppercase letters; vector variables with boldface lowercase letters, boldface uppercase calligraphic letters and boldface uppercase letters with a superscript; $\text{tr}(\mathbf{A})$ denotes the trace of the square matrix $\mathbf{A}$; $\mathbf{A}^v = \text{vec} (\mathbf{A})$ denotes a vector obtained stacking the columns of the matrix $\mathbf{A}$; the symbol $\otimes$ denotes the Kroneker product between two matrices; $|S|$ denotes the cardinality of the set $S$; $\mathbf{I}_M$ denotes the $M \times M$ identity matrix;

II. SYSTEM MODEL AND PILOT-AIDED CHANNEL ESTIMATION ALGORITHMS IN OFDM SYSTEM

According to the OFDM transmission scheme, the relation in the frequency domain between the transmitted and the received signals is given by [3] [4] [6]

$$Y[k,m] = X[k,m]H[k,m] + W[k,m]$$

where $Y[k,m]$ and $X[k,m]$ are the received and the transmitted signals, respectively, whereas $k$ is the index over the frequency carriers ($k = 0, 1, \ldots, K - 1$) and $m$ is the index over the OFDM symbols ($m = 0, 1, \ldots, M - 1$); the term $W$ is the AWGN noise with zero mean and variance $\sigma_w^2$, whereas $H$ represents the channel frequency response. We denote with $\mathbf{X}_p = \text{diag}(x_1, x_2, \ldots, x_p)$, the $P \times P$ matrix of transmitted pilot symbols, with $\mathcal{H}_p$ the channel frequency response at the pilot locations, and with $\mathcal{Y}_p$ the $P \times 1$ received signal vector at the pilot positions, respectively. An estimation of the channel frequency response over the pilot carriers can be easily achieved since the data transmitted by means of these carriers are known at the receiver: so, by multiplying $\mathcal{Y}_p$ for the matrix $\mathbf{X}_p^H$ we obtain the $P$-component vector $\hat{\mathcal{H}}_p$ of the estimated channel coefficients over pilot sub-carriers, that is [4] [6]

$$\hat{\mathcal{H}}_p = \mathbf{X}_p^H \mathcal{Y}_p = \mathcal{H}_p + \mathbf{X}_p^H \mathbf{W}_p$$

Pilot-aided channel estimation algorithms exploit this result to get an estimation on all the other sub-carriers through interpolation and estimation methods. Firstly, we will consider the interpolation algorithms that are based on a separable interpolation, where the interpolation procedures in the frequency and the time domains are performed separately [5]. Since the interpolation in the frequency domain is a linear operator and thanks to the knowledge of the vectors $\mathcal{H}_p$, we can express the interpolated pilot coefficient values as

$$\hat{\mathcal{H}}_f = \mathbf{A}_f \hat{\mathcal{H}}_p$$

where $\mathbf{A}_f$ represents a $K \times P$ matrix containing the interpolation coefficients ($\mathbf{A}_f$ is derived from the applied interpolation method and will not be presented for sake of conciseness). Afterwards, let $\hat{\mathcal{H}}_f$ be the matrix whose columns are the frequency-interpolated vectors $\mathcal{H}_f$, and $\mathbf{M}$ be the set of time indexes of the pilot-carrying OFDM symbols within an OFDM frame, then the matrix $\hat{\mathcal{H}}_f$ has dimension $K \times |\mathcal{M}|$ and can be expressed by

$$\hat{\mathcal{H}}_f = [\hat{\mathcal{H}}_f[^0] \hat{\mathcal{H}}_f[^1] \ldots \hat{\mathcal{H}}_f[^{|\mathcal{M}|}]]$$

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$$\hat{\mathcal{H}}_f = [\hat{\mathcal{H}}_f[^0] \hat{\mathcal{H}}_f[^1] \ldots \hat{\mathcal{H}}_f[^{|\mathcal{M}|}]]$$
where the index \(0 \leq i_k \leq |M| - 1\) denotes the position of the pilot-carrying OFDM symbols within the OFDM frame. Now, let \(\hat{H}_p\) be the \(P \times |M|\) matrix containing the estimated channel coefficients for the pilot symbols, that is

\[
\hat{H}_p = [\hat{H}_p[i_0] \ \hat{H}_p[i_1] \ \ldots \ \hat{H}_p[i_{|M|}]]
\]  

(5)

Making use of (3), a convenient form for expressing the content of \(\hat{H}_f\) is given by

\[
\hat{H}_f = \begin{pmatrix}
A_{f, i_0} & 0 & \ldots & 0 \\
0 & A_{f, i_1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_{f, i_{|M|}}
\end{pmatrix}
\]  

(6)

where \(A_{f,i}\) is a block diagonal matrix containing all the interpolation coefficients of matrix \(A_f\) for the pilot-carrying OFDM symbols. Consider now the application of the time-interpolation operator. Since it operates on column vectors, we have

\[
\hat{H}_f = [F_L \hat{h}_p[i_0] \ F_L \hat{h}_p[i_1] \ \ldots \ F_L \hat{h}_p[i_{|M|}]]
\]  

(15)

so that

\[
\hat{H}_f = F_1 \hat{h}_f^v
\]  

(16)

where \(\hat{h}_f^v\) is a vector containing the time-domain channel impulse responses estimated over the pilot-carrying OFDM symbols, \(F_L\) is the Fourier transform matrix which represents the DFT of the channel impulse response, and \(F_1 = I_{|M|} \otimes F_L\).

By applying the time-domain interpolation operator in (13), we achieve the interpolated channel coefficients for the whole sub-frame as

\[
\hat{H}^v = (B_t \otimes I_K) F_1 \hat{h}_f^v = B_1 \hat{h}_f^v
\]  

(17)

III. MSE COMPUTATION

The MSE analysis is a common way to measure the channel estimation performance, and to design the optimum pilot sequences and their position [4] [6] [8]. Let \(\Theta\) be a \(J\)-vector of parameters (in our case, the true channel coefficients) and let \(\hat{\Theta}\) be an estimate of \(\Theta\). The MSE between the true and the estimated variables is defined as

\[
\sigma^2 = \frac{1}{J} E[\|\Theta - \hat{\Theta}\|_2^2] = \frac{1}{J} \{\text{tr}[E(\Theta \Theta^H)] - \text{tr}[E(\Theta \hat{\Theta}^H)] - \text{tr}[E(\hat{\Theta} \Theta^H)] + \text{tr}[E(\hat{\Theta} \hat{\Theta}^H)]\}
\]  

(18)

The expression in (18) is usually used to compute the MSE for the pilot-carrying OFDM symbols [4] [6] [9]. The goal of this study is the derivation in a closed form of the MSE expression of a general OFDM system in the time-frequency domain. Therefore, to this aim, the variables \(\Theta\) and \(\hat{\Theta}\) has to contain the real channel and the estimated coefficients (exploiting the time domain interpolation) on the whole OFDM frame, respectively. Our computation method provides a general tool that can be applied to all linear channel estimation algorithms; moreover, it allows to obtain the channel estimation MSE for pilot symbols which are positioned according to a general lattice within the OFDM frame: as a result, it can be used as a support to the design process in order to obtain the best channel estimation algorithms, and to define the best pilot symbol patterns within the OFDM frame. In the next two sections we will present our MSE computation method for the two classes of considered algorithms.

A. Time-Frequency MSE evaluation for channel estimation algorithms based on separable interpolation

By using (14) and (18), and substituting \(\Theta = \hat{H}^v\) and \(\hat{\Theta} = A_{f,i} \hat{H}_f^v\) we obtain the total MSE

\[
\sigma^2 = \frac{1}{J} \{\text{tr}[A_{f,i} E(\hat{H}_f^v (\hat{H}_f^v)^H A_{f,i}] - A_{f,i} E(\hat{H}_f^v (\hat{H}_f^v)^H \text{tr}[E(\hat{H}_f^v (\hat{H}_f^v)^H)] \}
\]  

(19)
where $J = K \times M$.

Let $i_Q(p)$ and $j_Q(p)$ be the row-index and the column-index, respectively, of an element in a matrix $X$, having $Q$ rows, that occupies the $p$th position in $\text{vec}(X)$. We have

$$i_Q(p) = \text{rem}(p, Q)$$
$$j_Q(p) = \left\lfloor \frac{p}{Q} \right\rfloor$$

where $\text{rem}(p, Q)$ is the remainder of the division among the integers $p$ and $Q$.

From the definition of $H^v$ and $\hat{H}_p^v$ and of the theoretical frequency and time autocorrelation sequences $R_f$ and $R_t$ [4], we have

$$[E[H^v((H^v)^H)]_{k,l} = R_f[i_K(k) - i_K(l)]R_t[j_K(k) - j_K(l)]$$
$$[E[H^v((H^v)^H)]_{k,l} = R_f[i_K(k) - (i_P(l)D_f - D_{off}(j_P(l)))]$$
$$R_t[i_K(k) - D(j_P(l))]$$
$$[E[H_p^v((H_p^v)^H)]_{k,l} = R_f[(i_P(k) - i_P(l))D_f+]$$
$$D_{off}(j_P(l))]R_t[D(j_P(l)) - D(j_P(l))]$$
$$+ \sigma_p^2 \delta[k - l]$$
$$E[H_p^v((H_p^v)^H)] = E[H^v((H^v)^H)]^H$$

where the function $D(\cdot)$ maps the time index of a column of $\hat{H}_p^v$ onto the actual time position within the OFDM frame, and $D_{off}(\cdot)$ takes into account the possible offset of the first inserted pilot.

**B. Time-Frequency MSE evaluation for channel estimation algorithms based on channel impulse response estimation**

In order to apply (17) to MSE evaluation, a convenient form to express the true coefficients in the frequency domain for the whole OFDM frame is given by

$$H^v = F_2h^v$$

where vector $h^v$ contains the true time-domain channel impulse response coefficients of all the OFDM symbols in a frame, whereas $F_2 = I_M \otimes F_L$. From (18), by defining $\Theta = F_2h^v$ and $\hat{\Theta} = B_1\hat{h}_p^v$, the MSE on the whole frame can be expressed as

$$\sigma_t^2 = \frac{1}{J} \left\{ \text{tr}(B_1S_5E[h_p^v(h_p^v)^H]B_1^H - B_1E[\hat{h}_p^v(h_p^v)^H]F_2^H$$
$$- F_2E[\hat{h}_p^v(h_p^v)^H]B_1^H + F_2E[h^v(h^v)^H]F_2^H) \right\}$$

where $J = K \times M$.

In the case of LS/ML estimation we have

$$\hat{h}_p^v = h_p^v + S_4w_p^v$$

and equation (23) becomes

$$\sigma_t^2 = \frac{1}{J} \left\{ \text{tr}(B_1S_5E[h_p^v(h_p^v)^H]B_1^H - B_1E[\hat{h}_p^v(h_p^v)^H]F_2^H$$
$$+ B_1S_4E[w_p^v(w_p^v)^H]S_4^H B_1^H - F_2E[\hat{h}_p^v(h_p^v)^H]B_1^H$$
$$+ F_2E[h^v(h^v)^H]F_2^H) \right\}$$

where $w_p^v$ is the AWGN vector on the pilot-carrying OFDM symbols and $S_4 = I_{[M]} \otimes S_1$ with $S_1 = (S^H S)^{-1}S^H$, $S = \frac{1}{\sqrt{F}} F H A_p F_L$ and $F$ the complete Fourier transform matrix [6] [7].

Considering the case of MMSE estimation we have instead

$$\hat{h}_p^v = S_5h_p^v + S_6w_p^v$$

where $w_p^v$ is the AWGN vector on the pilot-carrying OFDM symbols and $S_4 = I_{[M]} \otimes S_2$ with $S_2 = V^{-1}S^H S$ and $S_3 = V^{-1}S^H$ [4] [6]. The correlation matrices $E[h_p^v(h_p^v)^H], E[\hat{h}_p^v(h_p^v)^H], E[h^v(h^v)^H]$ can be computed under the assumptions of uncorrelated channel path gains with time-varying coefficients, and their expressions is given by

$$E[h^v(h^v)^H]_{k,l} = PDP(i_L(k))\delta[i_L(k) - i_L(l)]$$
$$E[\hat{h}_p^v(h_p^v)^H]_{k,l} = PDP(i_L(k))\delta[i_L(k) - i_L(l)]$$
$$E[h_p^v(h_p^v)^H]_{k,l} = PDP(i_L(k))\delta[i_L(k) - i_L(l)]$$

where $i_{L}(k), j_{L}(k)$ are the frequency-time coordinates of the $k$th entry of either $h^v$ or $h_p^v$, with $L$ the channel length.

**IV. SIMULATION PARAMETERS**

In order to assess the validity of the derived MSE estimation method, we refer to the LTE-Advanced system. The downlink communication chain based on OFDM modulation has been implemented in a C++ simulator and several channel estimation algorithms, such as polynomial interpolation, LS, ML and MMSE methods have been compared. In order to verify the performance of the LTE-Advanced standard, two Multiple Input Multiple Output (MIMO) schemes have been taken into account, namely the $2 \times 2$ and the $4 \times 4$ Alamouti space-frequency coding schemes. All the parameters of the simulations are defined according to the LTE specifications for the 3MHz band: particularly, the LTE frame is composed by ten sub-frames. Each subframe is formed by two slots (14 OFDM symbols) with 15 physical resource blocks (PRB) each, where 120 or 60 pilots symbols are inserted in the first two antennas ($M = \{0, 4, 7, 11\}$) and in the third and in the fourth ones ($M = \{0, 8\}$) respectively [1]. All the others LTE parameters are reported in Table I. Moreover, according to the deployment of the LTE pilots, we have set $D_f = 6$ and $D_{off} = 3$. The simulations are performed in a macro-cell vehicular scenario with a multipath fading channel characterized by six Rayleigh distributed paths: in particular...
we have used the ITU vehicular A channel model, with classical Jake Doppler spectrum in order to take into account the time-variance of the channel [10]. As to the polynomial interpolation algorithms, in the simulation we have considered only linear and spline interpolations which demonstrate the best performance with respect to higher degree polynomials.

V. MSE THEORY VALIDATION AND ANALYSIS

First, we validate the MSE computation method which has been derived previously by comparing the theoretical values to the ones obtained from simulations. In Figure 1 and 2, we plot the theoretical MSE curves vs the Eb/N_0 for all the analysed algorithms along with the results of simulations. The methods are referred by indicating the algorithms which have been used for frequency and time estimation, e.g., Linear-Linear stands for linear interpolation in the frequency domain and linear interpolation in the time domain. As can be seen, the theoretical and the experimental curves are perfectly overlapped, thus validating the proposed model. The theoretical MSE curves, therefore, allow us to identify the best estimation method: from Figures 1 and 2 it is apparent that the MMSE-Linear estimator affords the best performance, whereas the worst one is obtained by Spline-Spline interpolation. As to polynomial interpolators, the best performance is afforded by Linear-Linear interpolation.

Moreover, the proposed method allows the MSE to be easily computed in a closed form for several channel models. In Figure 3, the MSE vs mobile terminal speeds ranging from 10 km/h to 250 km/h is depicted for three different estimators, by assuming Eb/N_0 = 10 dB (similar trends have been obtained for the other estimators and different Eb/N_0). In particular, the MSE remains constant as the Doppler frequency raises up: this behaviour is due to the appropriate LTE system design which defines a symbol time lower than the channel coherence interval. As demonstrated by Figure 3, the estimation method ranking does not change for different Doppler values; therefore, in the following we will present the results which are relative to f_d = 215 Hz (125 Km/h) case. In Figure 4 the BER curves for the best three estimator, namely MMSE-Linear, LS/ML-Linear and Linear-Linear estimators, are depicted for both 2 \times 2 and 4 \times 4 Alamouti schemes. As expected, the performance ranking encountered in the MSE analysis is confirmed in terms of BER performance. Specifically, the best performance is afforded by the MMSE-Linear method, which however needs the a-priori knowledge of \sigma_w^2. The LS/ML-Linear estimator demonstrates the second best performance; it is worth stressing that this method does not request the
knowledge of the noise variance: therefore, when this value is not known or estimated, the LS/ML-Linear estimator yields the best performance. The Linear-Linear method shows worst BER (as well as MSE) performance than MMSE-Linear and LS/ML-Linear techniques; nevertheless, it is characterised by the lowest complexity. As discussed in Section III our computation method is totally general and allows to change all the parameters which are related to the pilot positions within the OFDM frame, namely $i_k \in M$, $D_f$ and $D_{off}$, to compute the MSE and to compare the different pilot symbols patterns: particularly, we have tested some different sets for the possible time index $i_k$ while varying $D_{off}$ within $[0,4]$ in each pilot-carrying OFDM symbols. Figure 5 is related to the MSE which is obtained for the best estimator for two classes of algorithms, namely Linear-Linear interpolation and MMSE-Linear estimation. The results show that the best performance in all cases is obtained with $M = \{0, 6, 7, 13\}$ and $D_{off} = 3$. As we can see this configuration permit to obtain a better performance than the original LTE configuration.

VI. CONCLUSION

In this work we have developed a method to compute in a closed form the channel estimation MSE in the time-frequency domain for a general OFDM frame. The computation is based on equation (18) which is usually used to compute the MSE for the pilot-carrying symbols, whereas in this work it has been modified to take into account both frequency and time domain estimation. The computation method is completely general and holds for all linear channel estimation algorithms and for every choice of pilot symbol positions within the OFDM frame. Thanks to this method it is possible to find out the best channel estimation algorithm according to the given pilots lattice and to the considered scenario. In order to demonstrate its usefulness, we have compared its results considering the LTE-Advanced system in a vehicular scenario. The results have shown that the MMSE-Linear estimator achieves the best performance; moreover, by varying the pilot symbol patterns and comparing different OFDM configuration, we have shown that the proposed method can be used as a support in the design process.

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