# Blind Symbol Timing Recovery for Multi-Level/Multi-h CPM signals with High Spectral Efficiency

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***Abstract* — In this paper, symbol timing recovery for continuous phase modulated signals is investigated making use of a new method that exploits the received signal's cyclostationarity. This algorithm can be employed with either full or partial response signalling in the form of an open-loop estimator. Performance in terms of estimation error variance and bias is assessed by analysis and simulation.**

## I. Introduction

CONTINUOUS-PHASE modulations (CPMs) have been extensively studied in the literature for many years [1]. CPM was initially conceived for satellite applications, because of its very attractive property of constant envelope, but it found its first important practical application in wireless communication, becoming the standard scheme for GSM. In spite of its favourable features, current applications of CPM are limited to a few simple schemes because of implementation complexity and synchronization problems [2].

In this paper we consider timing synchronization for CPM signals. Symbol timing has to be recovered without any prior information about framing, fine carrier frequency and carrier phase, since the most efficient algorithms for those functions rely on symbol timing information. So, it is necessary to provide the receiver with blind timing information. For generalized MSK modulations one possible solution is to pass the in-phase component of the complex envelope through a nonlinearity such as a squaring device and then to extract by means of a PLL or a filter the clock synchronous periodic component contained by the output of the nonlinearity [3]. A different approach is to pass the IF signal through a second order nonlinearity to generate periodic components, at carrier and clock frequencies, that are extracted by a couple of PLLs [4]. Another approach is to make use of maximum likelihood estimation methods [5]. Lambrette and Meyr suggested two algorithms that allow efficient timing recovery for MSK based on the computation of the Fourier spectrum of the absolute value of the differential phases [6]. Another scheme that can also be used with generalized MSK modulations extracts clock reference by passing the sampled baseband waveform through the cascade of a nonlinearity, followed by a digital differentiatior whose average output represents the error signal to be employed in a tracking loop [7].

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The problem of blind timing recovery with linear modulations has several efficient solutions. In particular the workhorse is the Gardner's timing error detector to be used with a first or second order closed loop estimator [8]. Open loop symbol timing synchronization can also be carried out by means of the well-known Oerder-Meyr's [O-M] estimator [9]. Gini and Giannakis [10] showed that O-M estimator is a particular sample of a larger class of estimators that exploit the cyclostationarity properties of a data signal. We want to extend this analysis to CPM signals.

After this introduction, Section II introduces signal model and basic notations. Section III describes the timing estimation algorithm. Analytical and simulation results are compared in Section IV and some conclusions offered in Section V.

## II. Signal Analysis and Statement of the problem

The complex envelope of the received waveform is composed of signal plus noise

$$r(t) = s(t;\underline{a}) + w(t). \tag{1}$$

The noise is a complex-valued WGN and has independent real and imaginary components, $w_c(t)$ and $w_s(t)$ both bearing a double-sided power spectral density $N_0$. The signal can be written as

$$\begin{aligned} s(t;\underline{a}) &= \sqrt{2P}\exp\left\{ j\varphi(t,\underline{a})\right\} \\ &= \sqrt{(2E_s)/T}\exp\left\{ j2\pi h\sum_k a_k q(t-kT)\right\}, \end{aligned} \tag{2}$$

where $P$ is the signal power, $E_s$ represents the received energy per signalling interval, $T$ is the symbol period, and $\underline{a}$ is the vector of channel symbols on the relevant observation time interval. We assume that data symbols are equally likely, independent, and belong to an $M$-ary alphabet $\{\pm1, \pm3, ..., \pm(M-1)\}$. The parameter $h$ is the modulation index and $q(t)$ is the phase response of the modulator, which is related to the frequency response as follows

$$q(t) = \int_{-\infty}^{t} g(\alpha)d\alpha.$$

Pulse $g(t)$ is time limited to the interval $(0, LT)$, where $L$ is an integer called the correlation length, and it is normalized in such a way that $q(LT) = 1/2$. In this study two specific waveforms for $g(t)$ are used: rectangular $(L\text{REC})$ and raised-cosine $(L\text{RC})$ [1].

An important class of CPMs is the so-called "Multi-h Phase Coding". A multi-h phase coded signal can be obtained by frequency-modulating a carrier with cyclically varying modulation indexes. In this case the transmitted waveform is

$$s(t;\underline{a}) = \sqrt{(2E_s)/T}\,\exp\left\{ j2\pi\sum_k h_k a_k q(t-kT)\right\},$$

with $h_k$ modulation index over the symbol interval and $h_k = h_{k+H}$, where $H$ is the repetition period for the modulation indexes. The interval $HT$ is called superbaud period.

In the presence of a symbol timing offset the received signal is

$$r(t) = s(t-\delta;\underline{a}) + w(t). \tag{3}$$

The sensitivity of CPMs to a timing error depends quite noticeably on the signal format. Although error probability degradations due to synchronization imperfections are difficult to assess analytically, computer simulations show that for instance MSK (i.e. 1REC, $h = 1/2$, $M = 2$) is relatively insensitive to large timing offsets [2]. Other formats exhibit on the contrary larger sensitivity.

The sensitivity to symbol timing errors of 2REC, $M = 4$, $h = 1/5$ CPM is shown in Fig.1. The reason why we used 2REC, $h = 1/5$, $M = 4$ is related to power/spectral efficiency. When $h$ is small, the bandwidth of the signal is roughly proportional to the product $hM$, that represents the "peak" overall modulation index of the signal. If we want to keep the bandwidth efficiency constant, when the number of levels $M$ of the alphabet increases we also have to decrease the modulation index $h$, so that the product $hM$ stays approximately constant. The particular CPM scheme is chosen considering the spectral efficiency of a reference system, the 8-PSK modulation with a code-rate $2/3$. Looking for a CPM scheme with a 99% bandwidth equal to its 3-dB bandwidth, we obtain the outlined format.

## III. Cyclic-spectrum-based symbol timing recovery

### *A. Derivation of the algorithm*

As outlined in Section I, we try to extend to CPM the approach in [10] to symbol timing estimation, since those signals share with linear modulations the property of cyclostationarity.

Let's start with the computation of the (non-stationary) autocorrelation function of the CPM signal

$$\begin{aligned} R_s(t;\tau) &= E_a\left\{ s(t)s^*(t-\tau)\right\} \\ &= 2P\exp\left\{ j2\pi h\sum_k a_k\left[q(t-kT)-q(t-\tau-kT)\right]\right\} \\ &= 2P\prod_k \frac{\sin\left(2\pi hMp_\tau(t-kT)\right)}{M\sin\left(2\pi hp_\tau(t-kT)\right)}, \end{aligned} \tag{4}$$

where $p_\tau(t) \triangleq q(t) - q(t-\tau)$.

Pulse $p_\tau(t)$ is limited to the interval $[0, LT+\tau)$, so the $k$-th term in the product appearing in the previous equation is equal to 1 outside this interval.

This means that the product is indeed convergent and converges to a periodic function of $t$:

$$\rho_\tau(t) = \prod_k \frac{\sin\left(2\pi hMp_\tau(t-kT)\right)}{M\sin\left(2\pi hp_\tau(t-kT)\right)} = \rho_\tau(t+T).$$

This is shown in Fig. 2 for the simple case of MSK modulation. If we consider the unknown delay $\delta$ affecting the received signal $r(t)$, neglecting noise we have:

$$\begin{aligned} R_r(t;\tau) &= E_a\left\{ s(t-\delta)s^*(t-\delta-\tau)\right\} = 2P\rho_\tau(t-\delta) \\ &= 2P\prod_k \frac{\sin\left(2\pi hMp_\tau(t-\delta-kT)\right)}{M\sin\left(2\pi hp_\tau(t-\delta-kT)\right)}. \end{aligned} \tag{5}$$

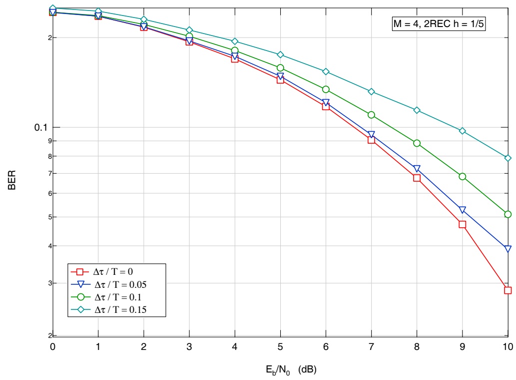


Fig. 1: Sensitivity of 2REC, $M = 4$, $h = 1/5$ to timing errors

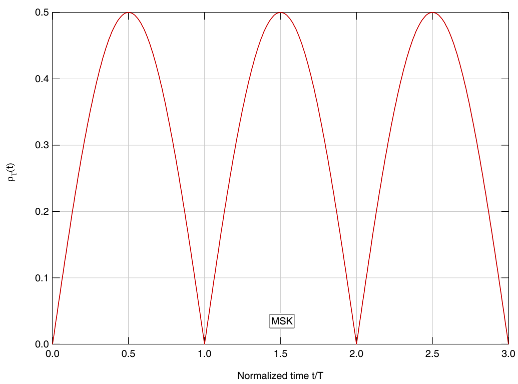


Fig. 2: Autocorrelation function

We expand now in a Fourier series wrt $t$ the autocorrelation function, obtaining

$$R_r(t;\tau) = 2P\sum_m C_m(\tau)e^{j2\pi m\frac{t}{T}}, \tag{6}$$

where

$$C_m(\tau) = \frac{1}{T}\int_0^T R_r(t;\tau)e^{-j2\pi m\frac{t}{T}}\,dt. \tag{7}$$

On the other hand, the Fourier coefficient for the undelayed CPM signal $s(t)$ is

$$C'_m(\tau) = \frac{1}{T}\int_0^T R_s(t;\tau)e^{-j2\pi m\frac{t}{T}}\,dt \tag{8}$$

so that

$$C_m(\tau) = C'_m(\tau) e^{-j2\pi m \frac{\delta}{T}}, \tag{9}$$

where $C'_m(\tau)$ is of course known in advance. Considering $m = \pm 1$ we have

$$C_{\pm 1}(\tau) = C'_{\pm 1}(\tau) e^{\mp j2\pi \frac{\delta}{T}} \tag{10}$$

which gives a first estimation algorithm

$$\hat{\delta} = \frac{T}{2\pi} \measuredangle C_{-1}(\tau) C'^{*}_{-1}(\tau). \tag{11}$$

This estimator is ambiguous by $T$ and insensitive to carrier phase. It requires real-time computation of the autocorrelation function (5) and of its Fourier coefficient (7).

*B. Performance evaluation and generalization to higher-order moments*

Now we want to carry out the performance evaluation of the algorithm. Assume that, when computing the autocorrelation function (5) in real time one can perform expectation on the data symbols only and not on noise. Then, neglecting in a first approximation the *noise x noise* interaction in the autocorrelation function computation, we have:

$$\begin{aligned} R_r(t;\tau) &= E_a\left\{ r(t) r^*(t-\tau) \right\} \\ &= E_a\left\{ \left[ s(t-\delta) + w(t) \right]\left[ s^*(t-\delta-\tau) + w^*(t-\tau) \right] \right\} \\ &\cong R_s(t-\delta;\tau) + s(t-\delta) w^*(t-\tau) + s^*(t-\delta-\tau) w(t) \\ &\triangleq R_s(t-\delta;\tau) + W'(t). \end{aligned} \tag{12}$$

Normalizing by $2P$ we get

$$R_r(t;\tau) = \rho_\tau(t-\delta) + W(t)$$

where $W(t) = W'(t)/(2P)$ is a noise term whose psd is constant and equal to $S_W(f) = 2N_0/P$.
Computing the Fourier coefficients we obtain

$$C_{-1,r}(\tau) = C'_{-1}(\tau) + W_1, \tag{13}$$

where $W_1$ is a complex-valued, independent-component random variable with zero mean and variance $\sigma^2_{w_1} = N_0/(PT) = 1/(E_s/N_0)$ on each component. We want to obtain the variance of the normalized estimate of the delay $\hat{\delta}/T$. We can write the estimation algorithm in this way

$$\frac{\hat{\delta}}{T} = \frac{1}{2\pi} \measuredangle \Omega$$

with

$$\begin{aligned} \Omega &= C_{-1,r}(\tau) C'^{*}_{-1,r}(\tau) = \left( C_{-1}(\tau) + W_1 \right)\left( C'^{*}_{-1}(\tau) + W_1^* \right) \\ &\cong \underbrace{C_{-1}(\tau) C'^{*}_{-1}(\tau)}_{\Psi} + \underbrace{C_{-1}(\tau) W_1^* + C'^{*}_{-1}(\tau) W_1}_{N} \\ &= \Psi + N = \Psi\left( 1 + \frac{N}{\Psi} \right). \end{aligned}$$

The first term on the right side of this equation is the value of the estimated parameter, so it doesn't contribute to the variance:

$$\frac{\hat{\delta}}{T} = \frac{\delta}{T} + \frac{1}{2\pi}\left[ \measuredangle\left( 1 + \frac{N}{\Psi} \right) \right].$$

At high SNR we have with high probability

$$|N/\Psi| << 1 \quad \Rightarrow \quad \measuredangle\left( 1 + \frac{N}{\Psi} \right) \sim \mathrm{Im}\left( \frac{N}{\Psi} \right)$$

so the variance is

$$\sigma^2_{\hat{\delta}/T} = \mathrm{var}\left\{ \frac{\hat{\delta}}{T} \right\} = \frac{1}{(2\pi)^2} E\left\{ \left[ \mathrm{Im}\left( \frac{N}{\Psi} \right) \right]^2 \right\} =$$

$$\frac{1}{(2\pi|\Psi|)^2} E\left\{ \left[ \mathrm{Im}(N) \right]^2 \right\} = \frac{\sigma_N^2}{(2\pi|\Psi|)^2} = \frac{\sigma_N^2}{(2\pi)^2 |C_{-1}(\tau)|^4}. \tag{14}$$

On the other hand

$$\sigma_N^2 = E\left\{ N^2 \right\} = 2\sigma^2_{W_1} |C_{-1}(\tau)|^2 = 2\frac{|C_{-1}(\tau)|^2}{E_s/N_0}$$

$$\sigma^2_{\hat{\delta}/T} \cong \frac{1}{2\pi^2} \frac{1}{E_s/N_0} \frac{1}{|C_{-1}(\tau)|^2} \text{ (High SNR)}. \tag{15}$$

Now we want to extend this approach to higher-order statistics of the CPM signal. Instead of computing the autocorrelation function, we investigate higher-order joint moments of the kind

$$R_s^{(k,l)}(t;\tau) = E_a\left\{ s^k(t) s^{*l}(t-\tau) \right\} \tag{16}$$

After simple computations we obtain

$$\begin{aligned} R_s^{(k,l)}(t;\tau) &= (2P)^{\left(\frac{k+l}{2}\right)} \prod_i \frac{\sin\left( 2\pi h M p_\tau^{(k,l)}(t-iT) \right)}{M \sin\left( 2\pi h p_\tau^{(k,l)}(t-iT) \right)} \\ &= (2P)^{\left(\frac{k+l}{2}\right)} \rho_s^{(k,l)}(t;\tau) \end{aligned} \tag{17}$$

where

$$p_\tau^{(k,l)}(t-iT) \triangleq kq(t) - lq(t-\tau).$$

The joint moment with $k = l$ always exists for any CPM signal. It can be shown that the general $(k,l)$ moment only exists when

$$\frac{\sin\left( 2\pi h M (k-l)/2 \right)}{M \sin\left( 2\pi h (k-l)/2 \right)} = 1. \tag{18}$$

A sufficient condition for (18) to hold is

$$k - l = m\gamma \tag{19}$$

where $\gamma$ is the denominator of the modulation index $h = \mu/\gamma$ and $m$ is an integer. When (19) is satisfied $R_s^{(k,l)}(t;\tau)$ is a periodic function in $t$ (Fig. 3), so that we can expand it in a Fourier series and we obtain an estimation algorithm such that obtained in the previous case. The expression for the variance is computed in the Appendix:

$$\sigma^2_{\hat{\delta}/T} \cong \frac{\left[\left(k^2+l^2\right)\right]}{4\pi^2} \frac{1}{E_s/N_0} \frac{1}{|C_{-1}(\tau)|^2}. \quad (20)$$

Fig. 4 represents the shape of $|C_{-1}(\tau)|$ as a function of $\tau$. It is clear from the expression of the variance that it is necessary to maximize the coefficient to minimize the variance. The amplitude of the Fourier coefficient is a function of the autocorrelation lag $\tau$. The search on $\tau$ has to be performed to find an optimum value for delay estimation.

*C. Generalization to Multi-h*

In the multi-h case the autocorrelation function is always

$$R_s(t;\tau) = E_a\left\{s(t)s^*(t-\tau)\right\} = 2P\prod_k \frac{\sin\left(2\pi h M p_\tau(t-kT)\right)}{M\sin\left(2\pi h p_\tau(t-kT)\right)}.$$

Bearing in mind the periodicity for the modulation indexes, we are able to put the autocorrelation function in the form

$$R_s(t;\tau) = 2P\prod_i \left\{\prod_{m=0}^{H-1} \frac{\sin\left(2\pi h M p_\tau\left(t-(m+iH)T\right)\right)}{M\sin\left(2\pi h p_\tau\left(t-(m+iH)T\right)\right)}\right\}. \quad (21)$$

It is still a periodic function of time, but the repetition period is $HT$, so we have to take into account this aspect in the Fourier expansion. We obtain the estimation algorithm

$$\frac{\hat{\delta}}{T} = \frac{H}{2\pi} \measuredangle C_{-1}(\tau) C'^{*}_{-1}(\tau), \quad (22)$$

where

$$C'_m(\tau) = \frac{1}{HT}\int_0^{HT} R_s(t;\tau) e^{-j2\pi m\frac{t}{HT}}\, dt \quad (23)$$

and

$$C_m(\tau) = \frac{1}{HT}\int_0^{HT} R_r(t;\tau) e^{-j2\pi m\frac{t}{HT}}\, dt\,. \quad (24)$$

The variance is

$$\sigma^2_{\hat{\delta}/T} \cong \frac{H^2}{2\pi^2} \frac{1}{E_s/N_0} \frac{1}{|C_{-1}(\tau)|^2}. \quad (25)$$

*D. Derivation of (Modified) Cramer-Rao Bound*

In this section we have described a new method for estimating a synchronization parameter. At this point we want to assess the ultimate accuracy that can be achieved in synchronization. The MCRB for timing recovery of a CPM signal bears a dependence on the response length of the modulation. It is found that [2]

$$MCRB(\delta/T) = \frac{1}{E_s/N_0} \frac{3L}{2\pi^2(M^2-1)h^2 N}$$

$$MCRB(\delta/T) = \frac{1}{E_s/N_0} \frac{L}{\pi^2(M^2-1)h^2 N},$$

for $L$REC and $L$RC respectively.

Considering the estimator (11) with its variance (15), we can derive an expression for the loss $\Delta$ in the two cases:

$$\Delta_{\mathrm{REC}} = 10\log\left(h^2\left(M^2-1\right)/(3L)\right) - 20\log\left(\left|C_{-1}(\tau)\right|\right)$$

$$\Delta_{\mathrm{RC}} = 10\log\left(h^2\left(M^2-1\right)/(2L)\right) - 20\log\left(\left|C_{-1}(\tau)\right|\right).$$

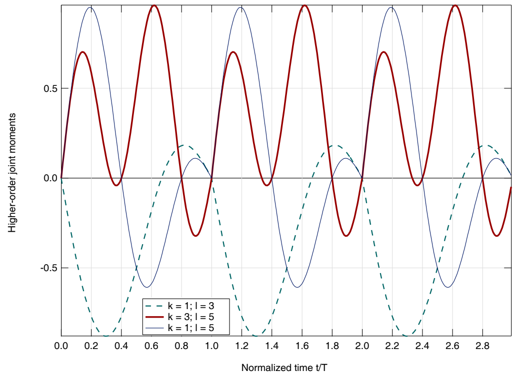


Fig. 3: High-order joint moments for MSK modulation

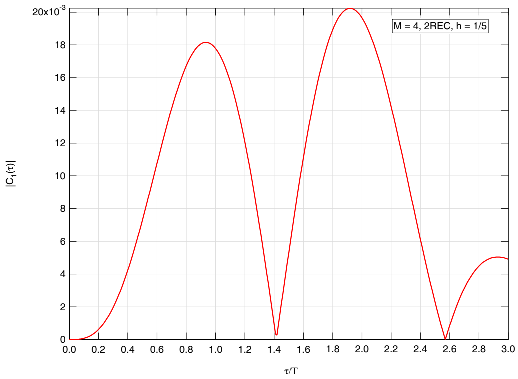


Fig. 4: Optimization of $\tau$ for 2REC, $M = 4, h = 1/5$

In the multi-h case the MCRB for timing estimation on an observation window of length $NHT$ is, for $L$RC pulses with easy generalisation to any pulse,

$$MCRB(\delta/T) = \frac{1}{E_s/N_0} \frac{L}{\pi^2(M^2-1)h_{eff}^2 NH},$$

where $h_{eff}^2 = \frac{1}{H}\sum_{i=0}^{H-1} h_i^2$ is the mean square modulation index.

In a real-world implementation, the autocorrelation function of the received signal is actually computed via a cyclic time-average instead of the data-only statistical expectation as in (12)

$$\hat{R}_r(t;\tau) = \frac{1}{N}\sum_{n=0}^{N-1} r(t-nT)r^*(t-\tau-nT), 0 \le t \le T.$$

Although in so doing the noise is not white any longer, we approximately have

$$\sigma^2_{\hat{\delta}/T} \cong \frac{1}{2\pi^2 N} \frac{1}{E_s/N_0} \frac{1}{|C_{-1}(\tau)|^2}$$

for (11),

$$\sigma_{\hat{\delta}/T}^2 \cong \frac{\left(k^2+l^2\right)}{4\pi^2 N}\frac{1}{E_s/N_0}\frac{1}{|C_{-1}(\tau)|^2}$$

in the higher order statistics case and

$$\sigma_{\hat{\delta}/T}^2 \cong \frac{H^2}{2\pi^2 N}\frac{1}{E_s/N_0}\frac{1}{|C_{-1}(\tau)|^2}$$

for multi-h signal.

### *E. Sensitivity to Frequency Offsets*

In this section we want to analyse the sensitivity of the algorithm to frequency offsets. We start, as usual, with the autocorrelation function, that in presence of a residual frequency offset $\nu$ becomes

$$\begin{aligned} R_r(t;\tau) &= E_a\left\{e^{j2\pi\nu(t-\delta)}s(t-\delta)e^{-j2\pi\nu(t-\delta-\tau)}s^*(t-\delta-\tau)\right\} \\ &= e^{j2\pi\nu\tau}2P\rho_\tau(t-\delta) \\ &= e^{j2\pi\nu\tau}2P\prod_k\frac{\sin\left(2\pi hMp_\tau(t-\delta-kT)\right)}{M\sin\left(2\pi hp_\tau(t-\delta-kT)\right)} \end{aligned}$$

The Fourier coefficients are

$$C_m(\tau) = C'_m(\tau)e^{-j2\pi m\frac{\delta}{T}}e^{j2\pi\nu\tau}$$

$$C_{\pm1}(\tau) = C'_{\pm1}(\tau)e^{\mp j2\pi\frac{\delta}{T}}e^{j2\pi\nu\tau}.$$

So the estimation algorithm is

$$\begin{aligned} \hat{\delta} &= \frac{T}{2\pi}\measuredangle\left[C_{-1}(\tau)C'^{\,*}_{-1}(\tau)\right] \\ &= \delta-\nu\tau T \end{aligned} \qquad (26)$$

We can derive the RMSE, defined as

$$RMSE = \sqrt{E\left\{\left(\frac{\hat{\delta}}{T}-\frac{\delta}{T}\right)^2\right\}}$$

This means that a residual frequency offset causes a timing estimation bias and an additional term equal to $(\nu\tau)^2$ to the root mean square estimation error

## IV. Numerical results

In this section we discuss the performance assessed by simulation of the estimators, expressed in terms of root mean square estimation error, normalized to the symbol period $\sigma_{\hat{\delta}/T}$.

Computer simulations have been run to check (15) and (20). Fig. 5 compares theoretical and simulations results for MSK modulation.

The corresponding MCRB is also shown for reference.

From Fig. 6 to Fig. 9 we show the same for others additional modulation formats, 3RC, $M=2$, $h=1/2$, 1RC, $M=4$, $h=1/2$, 2REC, $M=2, h=2/5$ and 2RC, $M=4, h=2/5$ respectively.

We recall that the theoretical line is obtained under the high-SNR assumption. Where this condition is not met the simulated curve diverges from the asymptotic one. On the contrary, for high SNRs the two curves diverge because of the effect of self-noise (typical of blind algorithms). The curves obtained making use of higher order statistics show better performance in almost all the considered cases, because the standard deviation of the estimate is lower.

In all the previous simulations the parameter $N$ is equal to 100 symbols. The results with $N=1000$ are shown in Fig.10. The rms error at 10 dB of SNR is close to $4 \cdot 10^{-3}$. Considering the sensitivity to timing errors, this should be good enough to give a negligible performance degradation.

Fig. 11 shows the results obtained for a particular multi-h scheme, 1RC, $M=2, h=[1/2, 3/4]$, in the case of an observation window of length $N=1000$.

## V. Conclusions

From the analysis above, some conclusions about our cyclostationarity-based estimator for symbol timing of a CPM signal can be drawn:

1) the algorithm is blind, feedforward and insensitive to carrier phase;
2) optimization can be easily performed to suit the algorithm to the particular signal that is considered by finding the optimum values of the autocorrelation lag and of the higher moments order;
3) the estimator suffers from self-noise at high SNR and noise enhancement at low SNR, but it is applicable to all practical CPM schemes and to multi-h case too;
4) a residual frequency offset brings a bias in the timing deviation evaluation.

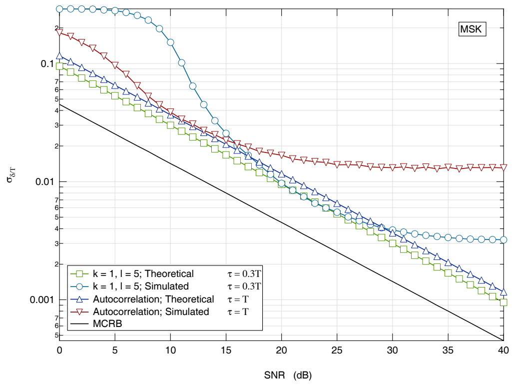


Fig. 5: Root-Mean Square Estimation Error for MSK modulation.

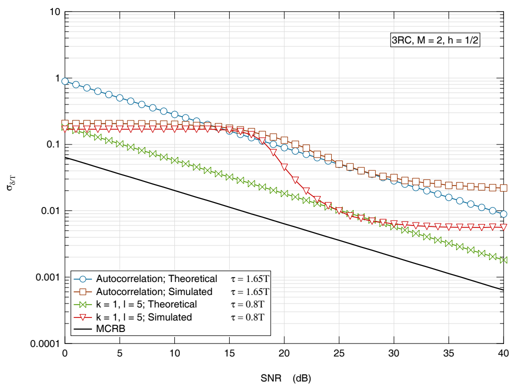


Fig. 6: Root-Mean Square Estimation Error for 3RC, $M=2, h=1/2$.

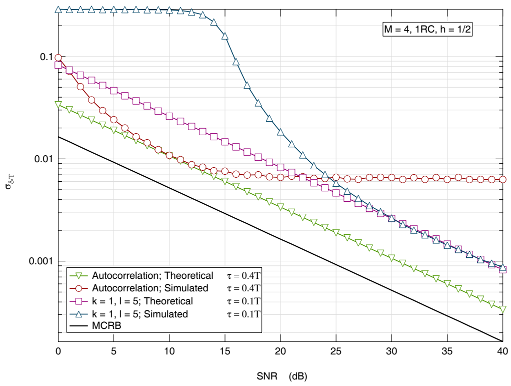


Fig. 7: Root-Mean Square Estimation Error for 1RC, $M=4, h=1/2$

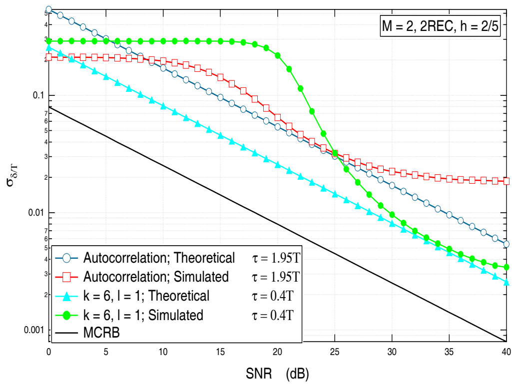


Fig. 8: Root-Mean Square Estimation Error for 2REC, $M=2, h=2/5$

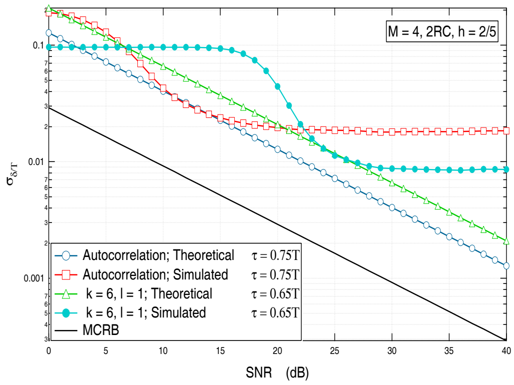


Fig. 9: Root-Mean Square Estimation Error for 2RC, $M=4, h=2/5$

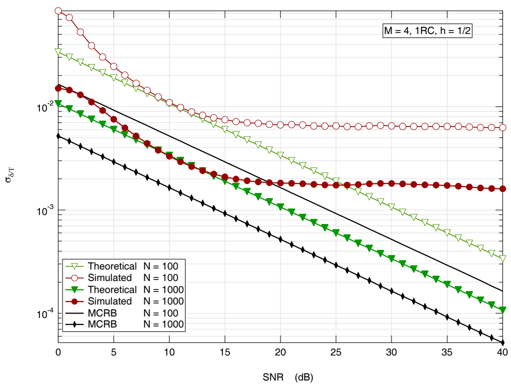


Fig. 10: 1RC, $M=4, h=1/2$, comparison between $N=100$ and $N=1000$.

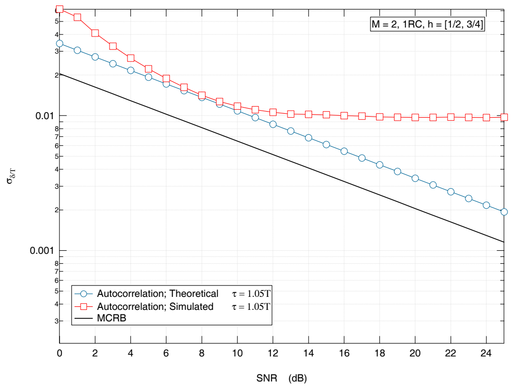


Fig. 11: -Mean Square Estimation Error for 1RC, $M=2, h=[1/2, 3/4]$, $N=1000$.

A digital implementation of the algorithm needs of course an over sampling factor wrt the symbol rate not to lose cyclostationarity. This aspect is still under investigation, together with the possibility of developing a data-aided version.
Performance still has to be optimized (order of joint moments).

## APPENDIX

In this Appendix we overview the passages leading to (20) in the text.

$$R_r^{(k,l)}(t;\tau) = E_a\left\{r^k(t)r^{*l}(t-\tau)\right\}$$
$$= E_a\left\{\left[s(t)+n(t)\right]^k\left[s(t-\tau)+n(t-\tau)\right]^{*l}\right\}.$$

We can expand in a Taylor series the power of the received signal

$$r^k(t) = \left[s(t)+n(t)\right]^k = \left\{s(t)\left(1+\frac{n(t)}{s(t)}\right)\right\}^k = s^k(t)\left(1+\frac{n(t)}{s(t)}\right)^k.$$

If the SNR is not too small, $|n(t)/s(t)|<1$, we can write

$$r^k(t) \cong s^k(t)\left(1+kn(t)/s(t)\right).$$

So we have

$$R_r^{(k,l)}(t;\tau) \cong E_a\left\{s^k(t)\left(1+k\frac{n(t)}{s(t)}\right)s^{*l}(t-\tau)\left(1+l\frac{n^*(t-\tau)}{s^*(t-\tau)}\right)\right\}$$
$$= E_a\left\{s^k(t)s^{*l}(t-\tau)\left[1+k\frac{n(t)}{s(t)}+l\frac{n^*(t-\tau)}{s^*(t-\tau)}\right]\right\}+$$
$$+E_a\left\{s^k(t)s^{*l}(t-\tau)\left[kl\frac{n(t)}{s(t)}\frac{n^*(t-\tau)}{s^*(t-\tau)}\right]\right\}.$$

Neglecting the *noise x noise* interaction we obtain

$$R_r^{(k,l)}(t;\tau) \cong E_a\left\{s^k(t)s^{*l}(t-\tau)+kn(t)s^{k-1}(t)s^{*l}(t-\tau)\right\}$$
$$+E_a\left\{ln^*(t-\tau)s^k(t)s^{*(l-1)}(t-\tau)\right\}.$$

At the end, normalizing by $(2P)^{((k+l)/2)}$, we have

$$R_r^{(k,l)}(t;\tau) \cong \rho_s^{(k,l)}(t;\tau) + W(t)$$
$$R_r^{(k,l)}(t;\tau) \rightleftharpoons C_{-1,r}(\tau) = C_{-1}(\tau) + W_{-1}$$

where $W_{-1}$ is a random variable with zero mean and variance

$$\sigma_{w_1}^2 = \left[\left(k^2 + l^2\right)\right] \Big/ \left(E_s / N_0\right).$$

We can obtain the variance of the estimate in the same way that leads to the (14) and in this case we have

$$\sigma_N^2 = \left[\left(k^2 + l^2\right)\right] |C_{-1}(\tau)|^2 \Big/ \left(E_s / N_0\right).$$

By substituting in the (14) we obtain the (20).

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